

An application of non-associative Composition-Diamond lemma*

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Abstract: In this paper, by using Gröbner–Shirshov bases for non-associative algebras invented by A. I. Shirshov in 1962, we show I. P. Shestakov’s result that any Akivis algebra can be embedded into its universal enveloping algebra.

Key words: non-associative algebra; Akivis algebra; universal enveloping algebra; Gröbner–Shirshov basis.

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1 Introduction

A. G. Kurosh [11] initiated to study free non-associative algebras over a field proving that any subalgebra of a free non-associative algebra is free. His student, A. I. Zhukov, proved in [20] that the word problem is algorithmically decidable in the class of non-associative algebras. Namely, he proved that word problem is decidable for any finitely presented non-associative algebra. A. I. Shirshov, also a student of Kurosh, proved in [15], 1953, that any subalgebra of a free Lie algebra is free. This theorem is now known as the Shirshov–Witt theorem (see, for example, [12]) for it was proved also by E. Witt [19]. Some later, Shirshov [16] gave a direct construction of a free (anti-) commutative algebra and proved that any subalgebra of such an algebra is again free (anti-) commutative algebra. Almost ten years later, Shirshov came back to, we may say, the Kurosh programme, and published two papers [17] and [18]. In the former, he gave a conceptual proof that the word problem is decidable in the class of (anti-) commutative non-associative algebras. Namely, he created the theory that is now known as Gröbner–Shirshov bases theory for (anti-) commutative non-associative algebras. In the latter, he did the same for Lie algebras (explicitly) and associative algebras (implicitly). Their main applications were the decidability of the word problem for any one-relater Lie algebra, the Freiheitsatz (the Freeness theorem) for Lie algebras, and the algorithm for decidability of the word problem for any finitely presented homogeneous Lie algebra. The same algorithm is valid for any finitely presented homogeneous associative algebra as well. Shirshov’s main technical

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discovery of [18] was the notion of composition of two Lie polynomials and implicitly two associative polynomials. Based on it, he gave the algorithm to construct the Gröbner–Shirshov basis for any ideal of a free Lie algebra. The same algorithm is valid in the associative case. This algorithm is in general infinite as well as, for example, Knuth–Bendix algorithm [10]. Shirshov proved that if a Gröbner–Shirshov basis of an ideal is recursive, then the word problem for the quotient algebra is decidable. It follows from Shirshov’s Composition–Diamond lemma that it is valid for free non-associative, free (anti-) commutative, free Lie and free associative algebras (see [17] and [18]). Explicitly the associative case was treated in the papers by L. A. Bokut [3] and G. Bergman [2].

Independently, B. Buchberger in his thesis (1965) (see [7]) created the Gröbner bases theory for the classical case of commutative associative algebras. Also, H. Hironaka in his famous paper [9] did the same for (formal or convergent) infinite series rather than polynomials. He called his bases as the standard bases. This term is used until now as a synonym of Gröbner (in commutative case) or Gröbner–Shirshov (in non-associative and non-commutative cases) bases.

There are a lot of sources of the history of Gröbner and Gröbner–Shirshov bases theory (see, for example, [8], [4], [5], [6]).

In the present paper we are dealing with the Composition–Diamond lemma for a free non-associative algebra, calling it as non-associative Composition–Diamond lemma. Shirshov mentioned it in [17] that all his results are valid for the case of free non-associative algebras rather than free (anti-) commutative algebras. For completeness, we prove this lemma in Section 2 in this paper. Then we apply this lemma to the universal enveloping algebra of an Akivis algebra giving an another proof of I. P. Shestakov’s result that any Akivis algebra is linear (see [13]).

An Akivis algebra is a vector space V over a field k endowed with a skew-symmetric bilinear product $[x, y]$ and a trilinear product (x, y, z) that satisfy the identity $[[x, y], z] + [[y, z], x] + [[z, x], y] = (x, y, z) + (z, x, y) + (y, z, x) - (x, z, y) - (y, x, z) - (z, y, x)$. These algebras were introduced in 1976 by M. A. Akivis [1] as tangent algebras of local analitic loops. For any (non-associative) algebra B one may obtain an Akivis algebra $Ak(B)$ by considering in B the usual commutator $[x, y] = xy - yx$ and associator $(x, y, z) = (xy)z - x(yz)$. Let $\{e_i\}_I$ be a basis of an Akivis algebra A . Then the nonassociative algebra $U(A) = M(\{e_i\}_I | e_i e_j - e_j e_i = [e_i, e_j], (e_i e_j) e_k - e_i (e_j e_k) = (e_i, e_j, e_k), i, j, k \in I)$ given by the generators and relations is the universal enveloping algebra of A , where $[e_i, e_j] = \sum_m \alpha_{ij}^m e_m$, $(e_i, e_j, e_k) = \sum_n \beta_{ijk}^n e_n$ and each $\alpha_{ij}^m, \beta_{ijk}^n \in k$. The linearity of A means that A is a subspace of $U(A)$ (see [13]). Remark also that any subalgebra of a free Akivis algebra is again free (see [14]).

2 Composition–Diamond lemma for non-associative algebras

Let $X = \{x_i | i \in I\}$ be a set, X^* the set of all associative words u in X , and X^{**} the set of all non-associative words (u) in X . Let k be a field and $M(X)$ be a k -space spanned by X^{**} . We define the product of non-associative words by the following way:

$$(u)(v) = ((u)(v)).$$

Then $M(X)$ is a free non-associative algebra generated by X .

Let I be a linearly ordered set. We order X^{**} by the induction on the length $|((u)(v))|$ of the words (u) and (v) in X^{**} :

- (i) If $|((u)(v))| = 2$, then $(u) = x_i > (v) = x_j$ iff $i > j$.
- (ii) If $|((u)(v))| > 2$, then $(u) > (v)$ iff one of the following cases holds:
 - (a) $|u| > |v|$.
 - (b) If $|u| = |v|$ and $(u) = ((u_1)(u_2))$, $(v) = ((v_1)(v_2))$, then $(u_1) > (v_1)$ or $((u_1) = (v_1) \text{ and } (u_2) > (v_2))$.

It is easy to check that the order “ $<$ ” on X^{**} is a monomial order in the following sense:

- (a) “ $<$ ” is a well order.
- (b) $(u) > (v) \implies (u)(w) > (v)(w) \text{ and } (w)(u) > (w)(v) \text{ for any } (w) \in X^{**}$.

Such an order is called deg-lex (degree-lexicographical) order and we use this order throughout this paper.

Given a polynomial $f \in M(X)$, it has the leading word $(\bar{f}) \in X^{**}$ according to the deg-lex order on X^{**} such that

$$f = \alpha(\bar{f}) + \sum \alpha_i(u_i),$$

where $(\bar{f}) > (u_i)$, $\alpha, \alpha_i \in k$, $(u_i) \in X^{**}$. We call (\bar{f}) the leading term of f . f is called monic if $\alpha = 1$.

Definition 2.1 Let $S \subset M(X)$ be a set of monic polynomials, $s \in S$ and $(u) \in X^{**}$. We define S -word $(u)_s$ by induction:

- (i) $(s)_s = s$ is an S -word of S -length 1.
- (ii) If $(u)_s$ is an S -word of S -length k and (v) is a non-associative word of length l , then

$$(u)_s(v) \text{ and } (v)(u)_s$$

are S -words of length $k + l$.

The S -length of an S -word $(u)_s$ will be denoted by $|u|_s$. Note that $\overline{(asb)} = (a\bar{s}b)$ if $(u)_s = (asb)$, where $a, b \in X^*$.

Let f, g be monic polynomials in $M(X)$. Suppose that there exist $a, b \in X^*$ such that $(\bar{f}) = (a(\bar{g})b)$. Then we set $(w) = (\bar{f})$ and define the composition of inclusion

$$(f, g)_{(w)} = f - (agb).$$

It is clear that

$$(f, g)_{(w)} \in Id(f, g) \text{ and } \overline{(f, g)_{(w)}} < (w).$$

Given a nonempty subset $S \subset M(X)$, we shall say that the composition $(f, g)_{(w)}$ is trivial modulo $(S, (w))$, if

$$(f, g)_{(w)} = \sum_i \alpha_i (a_i s_i b_i),$$

where each $\alpha_i \in k$, $a_i, b_i \in X^*$, $s_i \in S$, $(a_i s_i b_i)$ an S -word and $(a_i(\bar{s}_i)b_i) < (w)$. If this is the case, then we write $(f, g)_{(w)} \equiv 0 \pmod{(S, (w))}$. In general, for $p, q \in M(X)$, we write

$$p \equiv q \pmod{(S, (w))}$$

which means that $p - q = \sum \alpha_i (a_i s_i b_i)$, where each $\alpha_i \in k$, $a_i, b_i \in X^*$, $s_i \in S$, $(a_i s_i b_i)$ an S -word and $(a_i(\bar{s}_i)b_i) < (w)$.

Definition 2.2 Let $S \subset M(X)$ be a nonempty set of monic polynomials and the order “ $<$ ” as before. Then S is called a Gröbner-Shirshov basis in $M(X)$, if any composition $(f, g)_{(w)}$ with $f, g \in S$ is trivial modulo $(S, (w))$, i.e., $(f, g)_{(w)} \equiv 0 \pmod{(S, (w))}$.

Lemma 2.3 Let $(a_1 s_1 b_1)$, $(a_2 s_2 b_2)$ be S -words. If S is a Gröbner-Shirshov basis in $M(X)$ and $(w) = (a_1(\bar{s}_1)b_1) = (a_2(\bar{s}_2)b_2)$, then

$$(a_1 s_1 b_1) \equiv (a_2 s_2 b_2) \pmod{(S, (w))}.$$

Proof. We have $a_1 \bar{s}_1 b_1 = a_2 \bar{s}_2 b_2$ as associative words in the alphabet $X \cup \{\bar{s}_1, \bar{s}_2\}$. There are two cases to consider.

Case 1. Suppose that subwords \bar{s}_1 and \bar{s}_2 of w are disjoint, say, $|a_2| \geq |a_1| + |\bar{s}_1|$. Then, we can assume that

$$a_2 = a_1 \bar{s}_1 c \text{ and } b_1 = c \bar{s}_2 b_2$$

for some $c \in X^*$, and so, $w = (a_1(\bar{s}_1)c(\bar{s}_2)b_2)$. Now,

$$\begin{aligned} (a_1 s_1 b_1) - (a_2 s_2 b_2) &= (a_1 s_1 c(\bar{s}_2)b_2) - (a_1(\bar{s}_1)cs_2b_2) \\ &= (a_1 s_1 c((\bar{s}_2) - s_2)b_2) + (a_1(s_1 - (\bar{s}_1))cs_2b_2). \end{aligned}$$

Since $(\overline{(\bar{s}_2) - s_2}) < (\bar{s}_2)$ and $(\overline{s_1 - (\bar{s}_1)}) < (\bar{s}_1)$, we conclude that

$$(a_1 s_1 b_1) - (a_2 s_2 b_2) = \sum_i \alpha_i (u_i s_1 v_i) + \sum_j \beta_j (u_j s_2 v_j)$$

for some $\alpha_i, \beta_j \in k$, S -words $(u_i s_1 v_i)$ and $(u_j s_2 v_j)$ such that $(u_i(\bar{s}_1)v_i), (u_j(\bar{s}_2)v_j) < (w)$. Thus,

$$(a_1 s_1 b_1) \equiv (a_2 s_2 b_2) \pmod{(S, (w))}.$$

Case 2. Suppose that the subword \bar{s}_1 of w contains \bar{s}_2 as a subword. We assume that

$$(\bar{s}_1) = (a(\bar{s}_2)b), \quad a_2 = a_1 a \text{ and } b_2 = b b_1, \text{ that is, } (w) = (a_1 a(\bar{s}_2) b b_1)$$

for some S -word (as_2b) . We have

$$\begin{aligned} (a_1 s_1 b_1) - (a_2 s_2 b_2) &= (a_1 s_1 b_1) - (a_1 (as_2b) b_1) \\ &= (a_1(s_1 - (as_2b))b_1) \\ &= (a_1(s_1, s_2)_{(w_1)} b_1), \end{aligned}$$

where $(w_1) = (\bar{s}_1) = (a(\bar{s}_2)b)$. Since S is a Gröbner-Shirshov basis, $(s_1, s_2)_{(w_1)} = \sum_i \alpha_i (c_i s_i d_i)$ for some $\alpha_i \in k$, S -words $(c_i s_i d_i)$ with each $(c_i(\bar{s}_i)d_i) < (w_1) = (\bar{s}_1)$. Then,

$$\begin{aligned} (a_1 s_1 b_1) - (a_2 s_2 b_2) &= (a_1 (s_1, s_2)_{(w_1)} b_1) \\ &= \sum_i \alpha_i (a_1 (c_i s_i d_i) b_1) = \sum_j \beta_j (a_j s_j b_j) \end{aligned}$$

for some $\beta_j \in k$, S -words $(a_j s_j b_j)$ with each $(a_j(\bar{s}_j)b_j) < (w) = (a_1(\bar{s}_1)b_1)$. Thus,

$$(a_1 s_1 b_1) \equiv (a_2 s_2 b_2) \pmod{(S, (w))}. \quad \square$$

Lemma 2.4 *Let $S \subset M(X)$ be a subset of monic polynomials and $\text{Red}(S) = \{(u) \in X^{**} \mid (u) \neq (a(\bar{s})b), a, b \in X^*, s \in S \text{ and } (asb) \text{ is an } S\text{-word}\}$. Then for any $f \in M(X)$,*

$$f = \sum_{(u_i) \leq (\bar{f})} \alpha_i (u_i) + \sum_{(a_j(\bar{s}_j)b_j) \leq (\bar{f})} \beta_j (a_j s_j b_j),$$

where each $\alpha_i, \beta_j \in k$, $(u_i) \in \text{Red}(S)$ and $(a_j s_j b_j)$ an S -word.

Proof. Let $f = \sum_i \alpha_i (u_i) \in M(X)$, where $0 \neq \alpha_i \in k$ and $(u_1) > (u_2) > \dots$. If $(u_1) \in \text{Red}(S)$, then let $f_1 = f - \alpha_1 (u_1)$. If $(u_1) \notin \text{Red}(S)$, then there exist some $s \in S$ and $a_1, b_1 \in X^*$, such that $(\bar{f}) = (u_1) = (a_1(\bar{s}_1)b_1)$. Let $f_1 = f - \alpha_1 (a_1 s_1 b_1)$. In both cases, we have $(\bar{f}_1) < (\bar{f})$. Then the result follows from the induction on (\bar{f}) . \square

The proof of the following theorem is analogous to one in Shirshov [17]. For convenience, we give the details.

Theorem 2.5 *(Shirshov, Composition-Diamond for non-associative algebras) Let $S \subset M(X)$ be a nonempty set of monic polynomials and the order “ $<$ ” on X^{**} as before. Then the following statements are equivalent.*

- (i) S is a Gröbner-Shirshov basis.
- (ii) $f \in \text{Id}(S) \Rightarrow (\bar{f}) = (a(\bar{s})b)$ for some $s \in S$ and $a, b \in X^*$, where (asb) is an S -word.
- (ii)' $f \in \text{Id}(S) \Rightarrow f = \alpha_1 (a_1 s_1 b_1) + \alpha_2 (a_2 s_2 b_2) + \dots$, where $\alpha_i \in k$, $(a_1(\bar{s}_1)b_1) > (a_2(\bar{s}_2)b_2) > \dots$ and $(a_i s_i b_i)$ S -words.
- (iii) $\text{Red}(S) = \{(u) \in X^{**} \mid (u) \neq (a(\bar{s})b) a, b \in X^*, s \in S \text{ and } (asb) \text{ is an } S\text{-word}\}$ is a basis of the algebra $M(X|S)$.

Proof. (i) \Rightarrow (ii). Let S be a Gröbner-Shirshov basis and $0 \neq f \in \text{Id}(S)$. Then, we have

$$f = \sum_{i=1}^n \alpha_i (a_i s_i b_i),$$

where each $\alpha_i \in k$, $a_i, b_i \in X^*$, $s_i \in S$ and $(a_i s_i b_i)$ an S -word. Let

$$(w_i) = (a_i(\bar{s}_i)b_i), (w_1) = (w_2) = \dots = (w_l) > (w_{l+1}) \geq \dots$$

We will use the induction on l and (w_1) to prove that $(\overline{f}) = (a(\overline{s})b)$ for some $s \in S$ and $a, b \in X^*$.

If $l = 1$, then $(\overline{f}) = (\overline{a_1 s_1 b_1}) = (a_1(\overline{s_1})b_1)$ and hence the result holds. Assume that $l \geq 2$. Then, by Lemma 2.3, we have

$$(a_1 s_1 b_1) \equiv (a_2 s_2 b_2) \pmod{(S, (w_1))}.$$

Thus, if $\alpha_1 + \alpha_2 \neq 0$ or $l > 2$, then the result holds. For the case $\alpha_1 + \alpha_2 = 0$ and $l = 2$, we use the induction on (w_1) . Now, the result follows.

(ii) \Rightarrow (ii)'. Assume (ii) and $0 \neq f \in Id(S)$. Let $f = \alpha_1(\overline{f}) + \dots$. Then, by (ii), $(\overline{f}) = (a_1(\overline{s_1})b_1)$. Therefore,

$$f_1 = f - \alpha_1(a_1 s_1 b_1), \quad (\overline{f_1}) < (\overline{f}), \quad f_1 \in Id(S).$$

Now, by using induction on (\overline{f}) , we have (ii)'.

(ii)' \Rightarrow (ii). This part is clear.

(ii) \Rightarrow (iii). Suppose that $\sum_i \alpha_i(u_i) = 0$ in $M(X|S)$, where $\alpha_i \in k$, $(u_i) \in Red(S)$. It means that $\sum_i \alpha_i(u_i) \in Id(S)$ in $M(X)$. Then all α_i must be equal to zero. Otherwise, $\sum_i \alpha_i(u_i) = (u_j) \in Red(S)$ for some j which contradicts (ii).

Now, for any $f \in M(X)$, by Lemma 2.4, we have

$$f = \sum_{(u_i) \in Red(S), (u_i) \leq (\overline{f})} \alpha_i(u_i) + \sum_{(a_j(\overline{s_j})b_j) \leq (\overline{f})} \beta_j(a_j s_j b_j).$$

So, (iii) follows.

(iii) \Rightarrow (i). For any $f, g \in S$, by Lemma 2.4, we have

$$(f, g)_{(w)} = \sum_{(u_i) \in Red(S), (u_i) < (w)} \alpha_i(u_i) + \sum_{(a_j(\overline{s_j})b_j) < (w)} \beta_j(a_j s_j b_j).$$

Since $(f, g)_{(w)} \in Id(S)$ and by (iii), we have

$$(f, g)_{(w)} = \sum_{(a_j(\overline{s_j})b_j) < (w)} \beta_j(a_j s_j b_j).$$

Therefore, S is a Gröbner-Shirshov basis. \square

3 Gröbner-Shirshov basis for universal enveloping algebra of an Akivis algebra

In this section, we obtain a Gröbner-Shirshov basis for universal enveloping algebra of an Akivis algebra.

Theorem 3.1 Let $(A, +, [-, -], (-, -, -))$ be an Akivis algebra with a linearly ordered basis $\{e_i \mid i \in I\}$. Let

$$[e_i, e_j] = \sum_m \alpha_{ij}^m e_m, \quad (e_i, e_j, e_k) = \sum_n \beta_{ijk}^n e_n,$$

where $\alpha_{ij}^m, \beta_{ijk}^n \in k$. We denote $\sum_m \alpha_{ij}^m e_m$ and $\sum_n \beta_{ijk}^n e_n$ by $\{e_i e_j\}$ and $\{e_i e_j e_k\}$, respectively.

Let

$$U(A) = M(\{e_i\}_I \mid e_i e_j - e_j e_i = \{e_i e_j\}, (e_i e_j) e_k - e_i (e_j e_k) = \{e_i e_j e_k\}, i, j, k \in I)$$

be the universal enveloping algebra of A . Let

$$\begin{aligned} S = & \{f_{ij} = e_i e_j - e_j e_i - \{e_i e_j\} \ (i > j), \ g_{ijk} = (e_i e_j) e_k - e_i (e_j e_k) - \{e_i e_j e_k\} \ (i, j, k \in I), \\ & h_{ijk} = e_i (e_j e_k) - e_j (e_i e_k) - \{e_i e_j\} e_k - \{e_j e_i e_k\} + \{e_i e_j e_k\} \ (i > j, k \geq j)\}. \end{aligned}$$

Then

(i) S is a Gröbner-Shirshov basis for $U(A)$.

(ii) A can be embedded into the universal enveloping algebra $U(A)$.

Proof. (i). It is easy to check that

$$\overline{f_{ij}} = e_i e_j \ (i > j), \ \overline{g_{ijk}} = (e_i e_j) e_k \ (i, j, k \in I), \ \overline{h_{ijk}} = e_i (e_j e_k) \ (i > j, k \geq j).$$

So, we have only two kinds of compositions to consider:

$$(g_{ijk}, f_{ij})_{(e_i e_j) e_k} \ (i > j, j \leq k) \quad \text{and} \quad (g_{ijk}, f_{ij})_{(e_i e_j) e_k} \ (i > j > k).$$

For $(g_{ijk}, f_{ij})_{(e_i e_j) e_k}$, $(i > j, j \leq k)$, we have

$$\begin{aligned} & (g_{ijk}, f_{ij})_{(e_i e_j) e_k} \\ &= (e_j e_i) e_k - e_i (e_j e_k) + \{e_i e_j\} e_k - \{e_i e_j e_k\} \\ &\equiv -e_i (e_j e_k) + e_j (e_i e_k) + \{e_i e_j\} e_k + \{e_j e_i e_k\} - \{e_i e_j e_k\} \\ &\equiv 0. \end{aligned}$$

For $(g_{ijk}, f_{ij})_{(e_i e_j) e_k}$, $(i > j > k)$, by noting that, in A ,

$$\begin{aligned} & [[e_i, e_j], e_k] + [[e_j, e_k], e_i] + [[e_k, e_i], e_j] \\ &= (e_i, e_j, e_k) + (e_k, e_i, e_j) + (e_j, e_k, e_i) - (e_i, e_k, e_j) - (e_j, e_i, e_k) - (e_k, e_j, e_i), \end{aligned}$$

we have

$$\begin{aligned}
& (g_{ijk}, f_{ij})_{(e_i e_j) e_k} \\
&= (e_j e_i) e_k - e_i (e_j e_k) + \{e_i e_j\} e_k - \{e_i e_j e_k\} \\
&\equiv -e_i (e_j e_k) + e_j (e_i e_k) + \{e_i e_j\} e_k + \{e_j e_i e_k\} - \{e_i e_j e_k\} \\
&\equiv -e_i (e_k e_j) - e_i \{e_j e_k\} + e_j (e_i e_k) + \{e_i e_j\} e_k + \{e_j e_i e_k\} - \{e_i e_j e_k\} \\
&\equiv e_j (e_i e_k) - e_k (e_i e_j) - \{e_i e_k\} e_j + \{e_i e_j\} e_k - e_i \{e_j e_k\} \\
&\quad - \{e_k e_i e_j\} + \{e_i e_k e_j\} + \{e_j e_i e_k\} - \{e_i e_j e_k\} \\
&\equiv e_j (e_k e_i) + e_j \{e_i e_k\} - e_k (e_j e_i) - e_k \{e_i e_j\} - \{e_i e_k\} e_j + \{e_i e_j\} e_k \\
&\quad - e_i \{e_j e_k\} - \{e_k e_i e_j\} + \{e_i e_k e_j\} + \{e_j e_i e_k\} - \{e_i e_j e_k\} \\
&\equiv e_k (e_j e_i) + \{e_j e_k\} e_i + \{e_k e_j e_i\} - \{e_j e_k e_i\} + e_j \{e_i e_k\} - e_k (e_j e_i) - e_k \{e_i e_j\} \\
&\quad - \{e_i e_k\} e_j + \{e_i e_j\} e_k - e_i \{e_j e_k\} - \{e_k e_i e_j\} + \{e_i e_k e_j\} + \{e_j e_i e_k\} - \{e_i e_j e_k\} \\
&\equiv \{e_j e_k\} e_i - e_i \{e_j e_k\} + e_j \{e_i e_k\} - \{e_i e_k\} e_j + \{e_i e_j\} e_k - e_k \{e_i e_j\} \\
&\quad + \{e_k e_j e_i\} + \{e_i e_k e_j\} + \{e_j e_i e_k\} - \{e_j e_k e_i\} - \{e_k e_i e_j\} - \{e_i e_j e_k\} \\
&\equiv \{e_j e_k\} e_i - e_i \{e_j e_k\} + \{e_k e_i\} e_j - e_j \{e_k e_i\} + \{e_i e_j\} e_k - e_k \{e_i e_j\} \\
&\quad + \{e_k e_j e_i\} + \{e_i e_k e_j\} + \{e_j e_i e_k\} - \{e_j e_k e_i\} - \{e_k e_i e_j\} - \{e_i e_j e_k\} \\
&\equiv \{\{e_j e_k\} e_i\} + \{\{e_k e_i\} e_j\} + \{\{e_i e_j\} e_k\} \\
&\quad + \{e_k e_j e_i\} + \{e_i e_k e_j\} + \{e_j e_i e_k\} - \{e_j e_k e_i\} - \{e_k e_i e_j\} - \{e_i e_j e_k\} \\
&\equiv 0.
\end{aligned}$$

Thus, S is a Gröbner-Shirshov basis for $U(A)$.

(ii) follows from Theorem 2.5.

This completes our proof. \square

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